

Tilburg University

Basics of inventory management (Part 4)

de Kok, A.G.

Publication date:
1991

Document Version
Publisher's PDF, also known as Version of record

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):
de Kok, A. G. (1991). *Basics of inventory management (Part 4): The (s,S)-model*. (Research Memorandum FEW). Faculteit der Economische Wetenschappen.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

CBM

CBM
R

UNIVERSITY

KE

UNIVERSITEIT
BRABANT

7626

1991

523

POSTBOX 90153

5000 LE TILBURG

THE NETHERLANDS



* C I N O 1 9 2 3 *



DEPARTMENT OF ECONOMICS
RESEARCH MEMORANDUM



BASICS OF INVENTORY MANAGEMENT: PART 4
The (s,S)-model

A.G. de Kok

R9

FEW 523

653.42

BASICS OF INVENTORY MANAGEMENT: PART 4.

The (s,S) -model.

A.G. de Kok

Tilburg University
Department of Econometrics

and

Centre for Quantitative Methods
Lighthouse Consultancy
Eindhoven

Present address:

Philips Consumer Electronics
Logistics Innovation
Building SWA-516
P.O. box 80002
5600 JB EINDHOVEN
The Netherlands.

BASICS OF INVENTORY MANAGEMENT: INTRODUCTION

In the winter of 1989 the idea emerged to document the knowledge about inventory management models, that had been developed over almost 10 years of research and 5 years of practical applications in a number of consultancy projects. The main motivation to document the methodology underlying a number of well-proven algorithms was that most existing literature did not cover the practical applications encountered. Investigations revealed that most well-known algorithms were based on the assumptions of stable demand during lead times and large batch sizes. Both assumptions do not apply to the JIT environment characterized by short lead times and high order frequencies.

My starting point was the application of renewal theory to production-inventory models. It turned out that the same formalism was applicable to the classical inventory models, like periodic review and reorder point models. The attention of the analysis was focused on service levels and average inventories. The reason for this was that in many cases the problem was to find a relation between customer service requirements and holding costs for different planning scenarios. The algorithms developed turned out to be robust and fast.

The conviction grew that the methodology extended to most practically relevant service measures and to all classical inventory models. To be able to prove this sponsors were needed to provide the time and money to do the required research. The Catholic University Brabant and the Centre for Quantitative Methods accepted the research proposal. The result of the research is the series **Basics of Inventory Management**.

From the outset the objective was to develop a unified framework for all classical inventory models. It was important to relax a number of assumptions made in most literature. To the knowledge of the author for the first time arbitrary compound renewal demand processes are considered, thereby relaxing the assumption of Poisson customer arrival processes. This is very important in view of market concentrations (hyper markets, power retailers,

etc.). The outcome of the research should be a comprehensive set of algorithms, which can be used in practical situations, e.g. in inventory management modules of MRP and DRP packages.

In the course of the research the so-called PDF-method was developed, that provided a means to approximately solve all relevant mathematical equations derived in the analysis. The results of the approximation schemes were promising, yet under some conditions the performance was not adequate. Coincidentally, it turned out that the performance of the PDF-method deteriorated as the order batch size increased. In the area of large batch sizes other approximation schemes had already been developed, so that together with the PDF-method these algorithms covered the whole range of models.

Though starting from the idea to provide practically useful material to OR-practitioners, it soon turned out that the analysis required was quite detailed and mathematically intricate. Nonetheless I felt it necessary to document the derivations as well, since the analysis extends to other models than discussed in this series. The consequence of this choice is that the first 6 parts (c.q chapters) of this series are entirely mathematical. Yet the reader will find as a result of the analysis simple-to-use approximation schemes. To illustrate the applicability of the analysis, part VII is devoted to numerical analysis, part VIII compares the different inventory management models and part IX provides a number of practical cases.

Part I provides the background material from renewal theory and the PDF-method. Part II discusses the (R,S) -model, part III the (b,Q) -model and part IV the cost-optimal (s,S) -model. Based on the analysis in part II-IV we analyze in part V and VI the (R,b,Q) - and the (R,s,S) -model, respectively. A provisional list of references is given below.

I would like to thank Frank van der Duyn Schouten of the Catholic University Brabant for giving me the funds to do the research. The same holds for Jos de Kroon and Mynt Zijlstra from the Centre for Quantitative Methods of Philips. Furthermore, I would like to thank Marc Aarts and Jan-Maarten van Sonsbeek for programming work.

THE (s,S)-MODEL

In the (b,Q)-model we fixed the reorder quantity (or more precisely the reorder quantity must be a multiple of some minimal batch size), possibly taking into account transportation and handling characteristics. This reduces the flexibility of the stock keeping facility. We look for a inventory management policy which combines the continuous review capability of the (b,Q)-policy with the lot size flexibility of the (R,S)-policy. Such a policy is the (s,S)-policy. Under the (s,S)-policy inventory is managed as follows.

As soon as the inventory position drops below s an amount is ordered such that the inventory position is raised to S.

Usually the difference between s and S depends on holding costs and fixed ordering cost. The reorder level s depends on service level constraints or penalty costs. As with the (b,Q)-model we assume compound renewal demand, i.e.

A_n := the time between the arrival of the $(n-1)^{\text{st}}$ and n^{th} customer.

D_n := the demand of the n^{th} customer.

$\{A_n\}$ and $\{D_n\}$ are mutually independent sequences of independent identically distributed random variables. Lead times $\{L_n\}$ are identically distributed and we assume that orders arrive in the order of initiation, i.e. orders do not overtake.

Among the inventory management strategies the (s,S)-policy has been shown to be optimal. Optimality refers to minimization of order costs, holding costs and penalty costs. The reason why the (s,S)-policy is less practised is, that it is somewhat more difficult to implement from an organizational point of view than the (b,Q)-policy. We discuss the differences in costs associated with the (s,S)-policy and the (b,Q)-policy in chapter 8.

It should be noted that the (s,S)-policy and the (b,Q)-policy are identical if undershoots are negligible or fixed. This is the case for constant demand per customer or incremental demand at high

rate. Therefore we only discuss the case of non-negligible undershoots.

As in the previous chapters we concentrate on service measures (section 5.1.) and the mean physical stock. As a by-product we find an expression for the penalty costs assuming linear penalty cost per unit backordered per unit time. This is all discussed in section 5.2. Section 5.3. discusses a procedure that determines the (s,S) -policy that minimizes ordering, holding and penalty costs.

5.1. Service measures

Due to the fact that we have an order-up-to-level S the (s,S) -model is regenerative, whereas the (b,Q) -model was not. On the other hand the nice result that the inventory position is homogeneously distributed between the control levels no longer holds. Though there is great similarity between the (s,S) -model and the (b,Q) -model, these differences cause differences in the expressions for all performance characteristics.

At time 0 the reorder level s is undershot by an amount U_0 . At time σ_1 the reorder level s is undershot again by an amount U_1 . Due to the fact that the inventory position equals S after each undershoot, we have that the inventory position processes in consecutive order cycles are independent of each other: The (s,S) -model constitutes a regenerative inventory position process.

Using standard arguments we find that the net stock immediately after arrival of the order generated at time 0 equals $S-D[0,L_0]$. Then it is clear that the net stock immediately before arrival of the order generated at time σ_1 equals $S-D(0,\sigma_1+L_1]$.

The following equation is key to the analysis.

$$\begin{aligned} D(0,\sigma_1+L_1] &= D(0,\sigma_1] + D(\sigma_1,\sigma_1+L_1] \\ &= S-s+U_1 + D(\sigma_1,\sigma_1+L_1] \end{aligned} \tag{5.1}$$

We separate $D(0, \sigma_1 + L_1]$ into a fixed known part $S-s$, an undershoot U_1 for which we can apply the approximation from renewal theory, and a lead time demand $D(\sigma_1, \sigma_1 + L_1]$, which is independent of U_1 .

As before we concentrate on the P_2 - and \hat{P}_1 -service measures. Recall their definition.

P_2 := long-run fraction of demand satisfied directly from stock on hand.

\hat{P}_1 := long-run fraction of time the net stock is positive.

Let us first derive an expression for P_2 . Based on the evolution of the net stock over time, we find

$$P_2(s, \Delta) = 1 - \frac{E[(D(0, \sigma_1 + L_1) - S)^+] - E[(D(0, L_0) - (s + \Delta))^+]}{E[D(L_0, \sigma_1 + L_1)]},$$

where $\Delta = S-s$. Substitution of (5.1) and use of

$$D(L_0, \sigma_1 + L_1] = D(0, \sigma_1 + L_1] - D(0, L_0]$$

implies

$$P_2(s, \Delta) = 1 - \frac{E[(D(\sigma_1, \sigma_1 + L_1) + U_1 - s)^+] - E[(D(0, L_0) - (s + \Delta))^+]}{\Delta + E[U_1]} \quad (5.2)$$

It follows from the fact that both at time 0 and at time σ_1 an order is initiated that

$$D(0, L_0] \leq D(\sigma_1, \sigma_1 + L_1]$$

We develop expressions for $P_2(s, \Delta)$ based on two-moment approximations for U_1 and $D(0, L_0]$.

An expression for the pdf of U_1 depends on Δ .

(i) $\Delta = 0$

For the case of an (S,S)-control rule the undershoot of S is simply the demand of the arriving customer. Hence

$$P\{U_1 \leq x\} = F_D(x), \quad (5.3)$$

where D denotes the generic demand per customer.

(ii) For the case of Δ positive we approximate the pdf of U_1 by the pdf of the stationary residual lifetime distribution of the renewal process $\{D_n\}$. In order to yield a valid and accurate approximation we must assume

$$\begin{aligned} \Delta > E[D] \quad c_D^2 \leq 1 \\ \Delta > \frac{3}{2} c_D^2 E[D] \quad c_D^2 > 1 \end{aligned} \quad (5.4)$$

The lower bounds on Δ in (5.4) are based on extensive numerical experimentation (cf. De Kok [1987]). Provided condition (5.4) holds we claim that

$$P\{U_1 \leq x\} \approx \frac{1}{E[D]} \int_0^x (1 - \sigma_D(y)) dy \quad (5.5)$$

It is reasonable to state that cases (i) and (ii) cover all relevant cases. As soon as $\Delta \leq E[D]$ the (s,S)-policy operates more or less like an (S,S)-policy to which case (i) applies.

The first two moments of U_1 are easily derived from (5.3) and (5.5).

$$E[U_1] = \begin{cases} E[D] & \Delta=0 \\ \frac{E[D^2]}{2E[D]} & \Delta > LB \end{cases} \quad (5.6)$$

$$E[U_1^2] = \begin{cases} E[D^2] & \Delta=0 \\ \frac{E[D^3]}{3E[D]} & \Delta > LB \end{cases} \quad (5.7)$$

Here LB denotes the lower bound given by (5.4).

The first two moments of $D(0, L_0]$ are given in section 4.1. For sake of completeness we restate them here

$$E[D(0, L_0)] = \left(\frac{E[L]}{E[A]} + \frac{1}{2}(c_A^2+1) \right) E[D] \quad (5.8)$$

$$\begin{aligned} \sigma^2(D(0, L_0)) &= (c_A^2+c_D^2) \frac{E[L]}{E[A]} E^2[D] + \sigma^2(L) \frac{E^2[D]}{E^2[A]} \\ &+ \frac{(c_A^2-1)}{2} \sigma^2[D] + \frac{(1-c_A^4)}{12} E^2[D] \end{aligned} \quad (5.9)$$

Knowing the first two moments of U_1 and $D(0, L_0]$ we can apply the PDF-method. Define the pdf $\gamma(\cdot)$ by

$$\gamma(x) = P_2(x-\Delta, \Delta) \quad x \geq 0$$

Clearly $P_2(s, \Delta)=0$ when $s < -\Delta$, since in that case the inventory position is less than zero all the time and therefore the net stock, too. Then every demand is backordered.

As before let X_γ be the random variable associated with $\gamma(\cdot)$. Then we can calculate the first two moments of X_γ from

$$E[X_\gamma] = \int_0^{\infty} (1-\gamma(x)) dx$$

$$E[X_\gamma^2] = 2 \int_0^{\infty} x(1-\gamma(x)) dx$$

To get a flavour of the calculations involved in the derivation of $E[X_\gamma]$ and $E[X_\gamma^2]$ we elaborate on the derivation of $E[X_\gamma]$.

$$\begin{aligned}
 E[X_\gamma] &= \int_0^\infty (1 - P_2(x - \Delta, \Delta)) dx \\
 &= \int_0^\infty \frac{(E[(D(\sigma_1, \sigma_1 + L_1) + U - x + \Delta)^+] - E[(D(0, L_0) - x)^+]) dx}{\Delta + E[U_1]} \\
 &= \frac{1}{\Delta + E[U_1]} \int_0^\infty \int_0^\infty (y - (x - \Delta))^+ dF_{D(\sigma_1, \sigma_1 + L_1) + U_1}(y) dx \\
 &\quad - \frac{1}{\Delta + E[U_1]} \int_0^\infty \int_x^\infty (y - x) dF_{D(0, L_0)}(y) dx \\
 &= \frac{1}{\Delta + E[U_1]} \int_{-\Delta}^\infty \int_0^\infty (y - x)^+ dF_{D(\sigma_1, \sigma_1 + L_1) + U_1}(y) dx \\
 &\quad - \frac{1}{\Delta + E[U_1]} \int_0^\infty \int_0^y (y - x) dx dF_{D(0, L_0)}(y) \\
 &= \frac{1}{\Delta + E[U_1]} \int_{-\Delta}^0 \int_0^\infty (y - x) dF_{D(\sigma_1, \sigma_1 + L_1) + U_1}(y) dx \\
 &\quad + \frac{1}{\Delta + E[U_1]} \int_0^\infty \int_x^\infty (y - x) dF_{D(\sigma_1, \sigma_1 + L_1) + U_1}(y) dx \\
 &\quad - \frac{1}{\Delta + E[U_1]} \int_0^\infty \frac{1}{2} y^2 dF_{D(0, L_0)}(y) \\
 &= \frac{1}{\Delta + E[U_1]} \left\{ \Delta (E[D(\sigma_1, \sigma_1 + L_1) + U_1] + \frac{\Delta}{2}) + \frac{1}{2} E[(D(\sigma_1, \sigma_1 + L_1) + U_1)^2] \right. \\
 &\quad \left. - \frac{1}{2} E[D^2(0, L_0)] \right\}
 \end{aligned}$$

Rearrangement and gathering of terms yield

$$E[X_\gamma] = (D + E[D(0, L_0)]) + \frac{(E[U^2] - \Delta^2)}{2(\Delta + E[U])} \quad (5.10)$$

Without going into details we claim that $E[X_\gamma^2]$ can be found along the same lines as above to yield

$$\begin{aligned} E[X_\gamma^2] = \frac{1}{\Delta + E[U]} & \left\{ \frac{\Delta^2}{3} + (E[D(0, L_0)] + E[U])\Delta^2 \right. \\ & + (E[D^2(0, L_0)] + 2E[U]E[D(0, L_0)] + E[U^2])\Delta \\ & \left. + E[D^2(0, L_0)]E[U] + E[D(0, L_0)]E[U^2] + \frac{E[U^3]}{3} \right\} \end{aligned} \quad (5.11)$$

Since an (s, S)-policy operates as a (b, Q)-policy for negligible undershoots, (5.10) and (4.7) as well as (5.11) and (4.8) should coincide when assuming $U = 0$. This is easy to verify.

Equation (5.11) involves the third moment of U. To compute $E[U^3]$ we assume that U is gamma distributed. Hence

$$E[U^3] = (1 + c_U^2)(1 + 2c_U^2)E^3[U] \quad (5.12)$$

Hence c_U denoted the coefficient of variation of U.

Once we know $E[X_\gamma]$ and $E[X_\gamma^2]$ we can fit a gamma distributed $\hat{\gamma}(\cdot)$ to these two moments. Then we claim that

$$P_2(s, \Delta) \approx \hat{\gamma}(s + \Delta) \quad s \geq -\Delta$$

The inversion scheme described in chapter 2 can be applied to $\hat{\gamma}(\cdot)$ to obtain a solution of the following equation

$$P_2(s^*, \Delta) = \beta$$

We claim that

$$s^* \approx \hat{\gamma}^{-1}(\beta) - \Delta$$

Again we found a fast and accurate algorithm to find the reorder level s , such that the P_2 -service level equals some target value. The accuracy of the approximations resulting from applications of the PDF-method is ratified by the results in table 5.1.

Next we focus our attention on the \hat{P}_1 -measure. The analysis follows the derivation of approximation (4.33) for the \hat{P}_1 -measure in the (b, Q) -model.

We define $T^+(s, \Delta)$ analogously to $T^+(b, Q)$,

$T^+(s, \Delta) :=$ time the net stock is positive during the replenishment cycle $[I_0, \sigma_1 + I_1]$.

Then $\hat{P}_1(s, \Delta)$ is given by

$$\hat{P}_1(s, \Delta) = \frac{E[T^+(s, \Delta)]}{E[\sigma_1]} \quad (5.13)$$

An expression for $E[\sigma_1]$ can be obtained from renewal theoretic arguments. Let

$N :=$ the number of customers arriving in $(0, \sigma_1]$.

Then σ_1 can be written as

$$\sigma_1 = \sum_{n=1}^N A_n$$

Since N is independent of $\{A_n\}$ we have

$$E[\sigma_1] = E[N] E[A]$$

Furthermore we have that

$$\Delta + U_1 = \sum_{n=1}^N D_n,$$

since both sides of this equation describe the total demand in $(0, \sigma_1]$. Now N is a so-called stopping time for $\{D_n\}$ (cf. Çinlar [1975]), which implies

$$E \left[\sum_{n=1}^N D_n \right] = E[N] E[D]$$

(Note that N is not independent of $\{D_n\}$). Combining the above equations we find

$$E[\sigma_1] = \frac{(\Delta + E[U])}{E[D]} E[A] \quad (5.14)$$

It remains to find an expression for $E[T^+(s, \Delta)]$. As in section (4.1) we define $T^+(x, t)$ by

$T^+(x, t)$ = the time the net stock is positive during $(0, t]$, given the net stock at time 0 equals x , $x \geq 0$.

The analysis in section 2.3. yielded the basic result (2.53), which is repeated below.

$$\begin{aligned} E[T^+(x, t)] &= (E[\hat{A}] - E[A]) (1 - F_{D(0,t]}(x)) \\ &+ E[A] \left[M(x) - \int_0^x (M(x-y) dF_{D(0,t]}(x)) \right] \quad x \geq 0 \end{aligned} \quad (5.15)$$

Equation (5.15) assumes that time t is an arbitrary point in time. We assume that L_0 and $\sigma_1 + L_1$ are such arbitrary points in time.

Using the definitions of $T^+(s, \Delta)$ and $T^+(x, t)$ we find

$$E[T^+(s, \Delta)] = \int_0^\infty \int_0^{s+\Delta} E[T^+(s+\Delta-y, t)] dF_{D(0, L_0) | \tau_1 + L_1 - L_0 = t}(y) dF_{\tau_1 + L_1 - \tau_1}(t) \quad (5.16)$$

Combination of (5.15) and (5.16) yields after tedious algebra

$$\begin{aligned} E[T^+(s, \Delta)] = & \left(E[\hat{A}] - E[A] \right) (F_{D(0, L_0)}(s+\Delta) - F_{U_1 + D(\tau_1, \tau_1 + L_1)}(s)) \\ & + E[A] \left(\int_0^{s+\Delta} M(s+\Delta-y) dF_{D(0, L_0)}(y) \right. \\ & \left. - \int_0^s M(s-y) dF_{U_1 + D(\tau_1, \tau_1 + L_1)}(y) \right) \end{aligned} \quad (5.17)$$

Now we distinguish between the case of $\Delta=0$ and $\Delta>0$.

(i) $\Delta=0$

For this case $U_1 \triangleq D$. We apply the identity

$$M * F(x) = M(x) - 1 \quad x \geq 0$$

to the last integral of (5.17).

$$\int_0^s M(s-y) dF_{U_1 + D(\tau_1, \tau_1 + L_1)}(y) = \int_0^s (M(s-y) - 1) dF_{D(\tau_1, \tau_1 + L_1)}(y) \quad (5.18)$$

Substituting (5.18) into (5.17) leads us to

$$E[T^*(s, 0)] = (E[\tilde{A}] - E[A]) (F_{D(0, L_1)}(s) - F_{U_1 + D(\tau_1, \tau_1 + L_1)}(s)) + E[A] F_{D(\tau_1, \tau_1 + L_1)}(s) \quad (5.19)$$

Thus we have shown for the case of $\Delta=0$.

$$\hat{P}_1(s, 0) \approx \frac{(c_A^2 - 1)}{2} (F_{D(0, L_1)}(s) - F_{U_1 + D(0, L_1)}(s)) + F_{D(0, L_1)}(s) \quad (5.20)$$

(ii) $\Delta > 0$

Return to (5.17). The last integral on the right hand side can be simplified through the use of (4.30),

$$\int_0^s M(s-y) dF_{U_1 + D(\tau_1, \tau_1 + L_1)}(y) = \int_0^s \frac{(s-y)}{E[D]} dF_{D(\tau_1, \tau_1 + L_1)}(y)$$

This yields

$$E[T^*(s, \Delta)] = \left(E[\tilde{A}] - E[A] \right) (F_{D(0, L_1)}(s+\Delta) - F_{U_1 + D(\tau_1, \tau_1 + L_1)}(s)) + E[A] \left(\int_0^{s+\Delta} (M(s+\Delta-y) dF_{D(0, L_1)}(y) - \int_0^s \frac{(s-y)}{E[D]} dF_{D(\tau_1, \tau_1 + L_1)}(y) \right) \quad (5.21)$$

Equation (5.21) leaves us with a fundamental problem not encountered in the analysis of the (R,S) and (b,Q)-model. We cannot get rid of the renewal function $M(\cdot)$ in an elegant way, e.g. by convolving $M(\cdot)$ with $F_D(\cdot)$ or $F_U(\cdot)$.

At this particular point the power of the PDF-method surfaces most. There is no way expression (5.21) can be simplified, since in general there is no explicit expression for $M(\cdot)$. The PDF-method, however, approaches the problem in an indirect way.

Assuming a pdf associated with $\hat{P}_1(s, \Delta)$ only moments of the associated random variable are needed for the gamma fit. It happens to be that the PDF-method yields explicit expressions for the first two moments of the pdf associated with $\hat{P}_1(s, \Delta)$.

Let us first give the expression for $\hat{P}_1(s, \Delta)$ for the case of $\Delta > 0$ that follows from (5.21).

$$\begin{aligned} \hat{P}_1(s, \Delta) = & \frac{(c_A^2 - 1)E[D]}{2(\Delta + E[U])} (F_{D(0, L_0]}(s + \Delta) - F_{U+D(0, L_0]}(s)) \\ & + \frac{E[D]}{\Delta + E[U]} \left[\int_0^{s+\Delta} M(s + \Delta - y) dF_{D(0, L_0]}(y) - \int_0^s \frac{(s - y)}{E[D]} dF_{D(0, L_0]}(y) \right] \end{aligned} \quad (5.22)$$

Let $\gamma(\cdot)$ be the pdf associated with $\hat{P}_1(s, \Delta)$,

$$\gamma(x) = \hat{P}_1(x - \Delta, \Delta)$$

and let X_γ denote the random variable with pdf $\gamma(\cdot)$. For the case of $\Delta > 0$ we need tedious algebra and several limit theorems from renewal theory to obtain the expressions for $E[X_\gamma]$ and $E[X_\gamma^2]$. For the case of $\Delta = 0$ only routine calculations are required. We find the following,

$$\begin{aligned} E[X_\gamma] = & E[D(0, L_0)] - \frac{(c_A^2 - 1)}{2} E[D] & \Delta = 0 \\ & \frac{\Delta^2}{2(\Delta + E[U])} + E[D(0, L_0)] + \frac{(E[U^2] - 2E^2[U])}{2(\Delta + E[U])} & \Delta > 0 \end{aligned} \quad (5.23)$$

$$- \frac{(c_A^2 - 1)}{2} E[D]$$

$$\begin{aligned}
 E[X_\gamma^2] &= E[D^2(0, L_0)] - \frac{(c_A^2 - 1)}{2} (E[D^2] + 2E[D]E[D(0, L_0)]) \quad \Delta = 0 \\
 &\quad + \frac{\Delta^3}{3(\Delta + E[U])} + \frac{\Delta^2}{\Delta + E[U]} E[D(0, L_0)] + E[D^2(0, L_0)] \\
 &\quad + \frac{(E[U^2] - 2E^2[U])E[D(0, L_0)]}{\Delta + E[U]} \quad \Delta > 0 \quad (5.24) \\
 &\quad + \frac{(E[U^3] - 3E[U]E[U^2] + 3E^3[U])}{3(\Delta + E[U])} \\
 &\quad - \frac{(c_A^2)}{2} \left\{ E^2[D]E[D(0, L_0)] + \frac{(\Delta^2 + 2\Delta E[U] + 2E[U^2])E[D]}{\Delta + E[U]} \right\}
 \end{aligned}$$

To have some check on validity of these intricate expressions we compare $\hat{P}_1(s, \Delta)$ with $\hat{P}_1(b, Q)$ for the case of $U \equiv 0$. Then we find from (5.23) and (5.24) for the case of $\Delta > 0$.

$U \equiv 0, \Delta > 0$.

$$E[X_\gamma] = \frac{\Delta}{2} + E[D(0, L_0)] - \frac{(c_A^2 - 1)}{2} E[D] \quad (5.25)$$

$$\begin{aligned}
 E[X_\gamma^2] &= \frac{\Delta^2}{3} + \Delta E[D(0, L_0)] + E[D^2(0, L_0)] \\
 &\quad - \frac{(c_A^2 - 1)}{2} \{ 2E[D]E[D(0, L_0)] + \Delta E[D] \} \quad (5.26)
 \end{aligned}$$

Then indeed we find that (5.25) and (5.26) are identical to (4.34) and (4.35), respectively.

Though (5.23) and (5.24) are complicated expressions, they can considerably be simplified under the assumption of gamma distributed interarrival times, demand per customer and lead times.

This concludes the section on service measures. We found that the analysis of the P_2 -measure was as straightforward as with the

(R,S)-model and the (b,Q)-model. The analysis of the \hat{P}_1 -measure turned out to be quite complicated. Yet the PDF-method provided the means to obtain explicit expressions for the moments of the random variable associated with the \hat{P}_1 -measure.

5.2. Physical stock and backlog

Section 5.1. provided us the means to compute the reorder-level that yields the required service given the minimal order size Δ . We are still interested in the amount of capital tied up in stocks. Towards this end we derive an approximation for the mean physical stock under the (s,S)-regime. It will turn out that the derivation of the results needed is far more complicated than with the (R,S)-model or (b,Q)-model. We saw the same thing happen with the \hat{P}_1 -measure. Readers only interested in the results should skip section 5.2.1. and 5.2.2.

5.2.1. Exploring the relation between backlog and physical stock

In this section we derive an exact expression for the mean backlog, which will be convenient for further calculation. Analogously to the analysis for the (b,Q)-model we find that

$$E[X^*(s,\Delta)] = E[Y(s,\Delta)] - E[O] + E[B(s,\Delta)], \quad (5.27)$$

where

$E[Y(s,\Delta)]$:= the mean inventory position.

$E[B(s,\Delta)]$:= the mean backlog.

The cost arguments that yield (3.53) now yield

$$E[O] = E[D] \frac{E[L]}{E[A]} \quad (5.28)$$

as in the (b,Q)-model.

To find an expression for $E[Y(s,\Delta)]$ we proceed as follows. Assume that the stock keeping facility incurs a cost of \$1 per item on stock.

Define $k_1(.)$ as

$k_1(x) :=$ total expected cost incurred until the inventory position drops below 0, given that at time 0 the inventory position equals $x \geq 0$ and no orders are initiated after time 0.

Then it is easy to see that

$$E[Y(s,\Delta)] = s + \frac{1}{E[\tau_1]} k_1(\Delta) \quad (5.29)$$

In chapter 2 (cf. also De Kok [1987]) we derived an exact expressions for $k_1(.)$,

$$k_1(x) = \frac{E[A]}{E[D]} \left\{ \frac{x^2}{2} - \frac{E[U^2(x)]}{2} + \frac{E[D^2]}{2E[D]} (x + E[U(x)]) \right\} \quad (5.30)$$

where $U(.)$ is the undershoot of 0 at the time the inventory position drops below 0. In general we do not have exact expressions for the moments of $U(.)$, yet we have already seen that the stationary residual life time provides a good approximation if Δ is not too small.

Combination of (5.27) - (5.30) yields

$$E[X^*(s,\Delta)] = s + \frac{1}{\Delta + E[U(\Delta)]} \left\{ \frac{\Delta^2}{2} - \frac{E[U^2(\Delta)]}{2} + \frac{E[D^2]}{2E[D]} (\Delta + E[U(\Delta)]) \right\} - E[D] \frac{E[L]}{E[A]} + E[B(s,\Delta)] \quad (5.31)$$

where we substituted equation (5.14) for $E[\sigma_1]$.

Next we give an approximate expression for $E[X^+(s, \Delta)]$ which is based on the approximation for the function $k(x, t)$ defined in chapter 2.

The analysis is analogue to the analysis of $E[X^+(b, Q)]$. Then we obtain after lengthy algebra

$$\begin{aligned}
 E[X^+(s, 0)] &\approx \frac{(c_A^2 - 1)}{2} \left[\int_0^s (s-y) dF_{D(0, L_q)}(y) - \int_0^s (s-y) dF_{D+D(0, L_q)}(y) \right] \\
 &\quad + \int_0^s (s-y) dF_{D(0, L_q)}(y) \\
 E[X^+(s, \Delta)] &\approx \frac{(c_A^2 - 1)}{2} \frac{E[D]}{\Delta + E[U]} \left\{ \Delta + E[U] + \int_{s+\Delta}^{\infty} (y-s-\Delta) dF_{D(0, L_q)}(y) \right. \\
 &\quad \left. - \int_s^{\infty} (y-s) dF_{U+D(0, L_q)}(y) \right\} \quad \Delta > 0 \quad (5.32) \\
 &\quad + \frac{E[D]}{(\Delta + E[U])} \left[\int_0^{s+\Delta} \int_0^{s+\Delta-y} (s+\Delta-y-z) dM(z) dF_{D(0, L_q)}(y) \right. \\
 &\quad \left. - \int_0^s \frac{(s-y)^2}{2E[D]} dF_{D(0, L_q)}(y) \right]
 \end{aligned}$$

Here U is the stationary residual life time associated with the renewal process $\{D_n\}$. The expression for $E[X^+(s, \Delta)]$ for $\Delta=0$ can be routinely calculated. It also permits application of the PDF-method. The results of the application of the PDF-method are postponed until we finished the analysis of the case $\Delta>0$. So let us for the moment assume that $\Delta>0$.

Remark: Approximation (5.32) can be applied directly to yield an approximation for $E[X^+(s, \Delta)]$ when we use the following limit theorem, which holds for any random variable X .

$$\lim_{x \rightarrow \infty} \int_0^x \int_0^{x-y} (x-y-z) dM(z) dF_X(y) \\ - \left[\frac{x^2}{2E[D]} + \left[\frac{E[D^2]}{2E^2[D]} - \frac{E[X]}{E[D]} \right] x \right. \\ \left. + \frac{E[X^2]}{2E[D]} - \frac{E[D^3]}{6E^2[D]} + \frac{E^2[D^2]}{4E^3[D]} - \frac{E[X]E[D^2]}{2E^2[D]} \right] = 0$$

Assuming s sufficiently large, we obtain after some algebra

$$E[X^*(s, \Delta)] \approx \frac{(C_A^2 - 1)}{2} \frac{E[D]}{\Delta + E[U]} \left\{ \Delta + E[U] + \int_{s-\Delta}^{\infty} (y - s - \Delta) dF_{D(0, L_0)}(y) \right. \\ \left. - \int_s^{\infty} (y - s) dF_{U+D(0, L_0)}(y) \right\} \quad \Delta > 0 \quad (5.33) \\ + s - E[D(0, L_0)] + E[U] \\ + \frac{\Delta^2 - E[U^2]}{2(\Delta + E[U])} - \int_s^{\infty} \frac{y - s^2}{2(\Delta + E[U])} dF_{D(0, L_0)}(y)$$

The above expression involves only one integral with which we did not deal before. One might decide to apply the PDF-method to get rid of these integrals. We have tested this approximation by fitting mixtures of Erlang distributions to $D(0, L_0)$ and then explicitly elaborating the integrals. The performance of approximation (5.33) is quite good provided s reasonably large, i.e. P_2 -level associated with $s \geq 0.7$.

Now we combine equations (5.31) and (5.32). Considerable algebra reveals that

$$\begin{aligned}
 E[B(s, \Delta)] &= \frac{(c_A^2 - 1)}{2} \frac{E[D]}{\Delta + E[U]} \left[\int_{s+\Delta}^{\infty} (Y - s - \Delta) dF_{D(0, L_0]}(Y) \right. \\
 &\quad \left. - \int_s^{\infty} (Y - s) dF_{U+D(0, L_0]}(Y) \right] \\
 &+ \frac{E[D]}{(\Delta + E[U])} \left[\int_0^{s+\Delta} \int_0^{s+\Delta-y} (s + \Delta - y - z) dM(z) dF_{D(0, L_0]}(Y) \right. \\
 &\quad \left. - a_2(s + \Delta)^2 - a_1(s + \Delta) - a_0 \right] \\
 &+ \frac{1}{2(\Delta + E[U])} \int_s^{\infty} (Y - s)^2 dF_{D(0, L_0]}(Y)
 \end{aligned} \tag{5.34}$$

The constant a_2 , a_1 and a_0 are given by

$$a_2 = \frac{1}{2E[D]}$$

$$a_1 = \frac{E[D^2]}{2E^2[D]} - \frac{E[D(0, L_0)]}{E[D]}$$

$$a_0 = \frac{E[D^2(0, L_0)]}{2E[D]} - \frac{E[D^2]}{6E^2[D]} + \frac{E^2[D^2]}{4E^2[D]} - E[D(0, L_0)] \frac{E[D^2]}{2E^2[D]}$$

We seem to have made hardly any progress when comparing equation (5.32) with equation (5.34). Yet there is an essential difference. We know that

$$\lim_{s \rightarrow \infty} E[B(s, \Delta)] = 0$$

and $E[B(s, \Delta)]$ monotone decreasing in s . This suggests application of the PDF-method, which applies not only for high values of s (as with (5.33)), but for any value of s . Indeed we apply the PDF-method to $E[B(s, \Delta)]$.

5.2.2. The PDF-method applied to the mean backlog

The mean backlog is a service measure. However, unlike the \hat{P}_1 and P_2 -measure it does not constitute a pdf, since the mean backlog approaches infinity as s approaches minus infinity. Therefore we have to apply a normalization.

As with the \hat{P}_1 - and P_2 -measure we only consider values of $s \geq -\Delta$ in our PDF-analysis. Before doing so we derive an expression for $B(s, \Delta)$ when $s \leq -\Delta$, though this may not be practically relevant. If $s \leq -\Delta$ then $E[X^+(s, \Delta)] = 0$. Then (5.32) reduces to

$$E[B(s, \Delta)] = E[D] \frac{E[L]}{E[A]} - s - E[U] - \frac{(\Delta^2 - E[U^2])}{2(\Delta + E[U])} \quad s \leq -\Delta \quad (5.35)$$

It follows from (5.35) that

$$E[B(-\Delta, \Delta)] = E[D] \frac{E[L]}{E[A]} + \Delta - E[U] - \frac{(\Delta^2 - E[U^2])}{2(\Delta + E[U])} \quad (5.36)$$

We know that $E[B(s, \Delta)]$ is monotone decreasing in s and approaches zero when s tends to infinity. By normalizing $E[B(s, \Delta)]$ by dividing it by $E[B(-\Delta, \Delta)]$, we can create a pdf $\gamma(\cdot)$,

$$\gamma(x) := 1 - \frac{B(x - \Delta, \Delta)}{B(-\Delta, \Delta)}, \quad x \geq 0 \quad (5.37)$$

As before we define X_γ the random variable associated with $\gamma(\cdot)$. Then we have that

$$E[X_\gamma] = \int_0^\infty \frac{B(x - \Delta, \Delta)}{B(-\Delta, \Delta)} dx \quad (5.38)$$

$$E[X_\gamma^2] = 2 \int_0^\infty x \frac{B(x-\Delta, \Delta)}{B(-\Delta, \Delta)} dx \quad (5.39)$$

Considering approximation (5.34) for $E[B(s, \Delta)]$ we conclude that (5.38) and (5.39) can be routinely calculated from previously obtained results once we know the following integrals.

$$I_1 := \int_0^\infty \left\{ \int_0^x \int_0^{x-y} (x-y-z) dM(z) dF_{D(0, L_0)}(y) - a_2 x^2 - a_1 x - a_0 \right\} dx$$

$$I_2 := \int_0^\infty x \left\{ \int_0^x \int_0^{x-y} (x-y-z) dM(z) dF_{D(0, L_0)}(y) - a_2 x^2 - a_1 x - a_0 \right\} dx$$

It is not at all clear that both I_1 and I_2 exist, since in both integrals we subtract two parts which diverge for x large. Luckily both integrals do exist. Computing I_1 and I_2 is not a trivial matter and requires derivation of several limit theorems from renewal theory. We only present the final results.

$$\begin{aligned} I_1 = & \frac{-E[D^3(0, L_0)]}{6E[D]} + \frac{E[D^2]}{4E^2[D]} E[D^2(0, L_0)] - \left(\frac{E^2[D^2]}{4E^3[D]} - \frac{E[D^3]}{6E^2[D]} \right) E[D(0, L_0)] \\ & + \frac{E[D^4]}{24D^2[D]} - \frac{E[D^2]E[D^3]}{6E^3[D]} + \frac{E^3[D^2]}{8E^4[D]} \end{aligned} \quad (5.40)$$

$$\begin{aligned}
 I_2 = & \frac{-E[D^4(0, L_0)]}{24E[D]} + \frac{E[D^2]}{12E^2[D]} E[D^3(0, L_0)] \\
 & - \left(\frac{E^2[D^2]}{12E^3[D]} - \frac{E[D^3]}{12E^2[D]} \right) E[D^2(0, L_0)] \\
 & - \left(\frac{E[D^3]E[D^2]}{12E^2[D]} - \frac{E[D^4]}{24E^2[D]} - \frac{E^3[D^2]}{8E^4[D]} \right) E[D(0, L_0)] \quad (5.41) \\
 & + \frac{E[D^5]}{120E[D]} - \frac{E[D^4]E[D^2]}{24E^3[D]} - \frac{E^2[D^3]}{36E^3[D]} + \frac{E[D^3]E^2[D^2]}{18E^4[D]} \\
 & - \frac{E^4[D^2]}{16E^5[D]}
 \end{aligned}$$

Clearly, I_1 and I_2 are complex expressions. Yet assuming gamma distributed demand the expressions can be considerably simplified. Also I_1 and I_2 express the dependence of approximations via the PDF-method on the moments of D and $D(0, L_0)$.

Substituting I_1 and I_2 at the appropriate places in (5.38) and (5.39) we find from (5.34),

$$\begin{aligned}
 E[X_\gamma] = & \frac{1}{E[B(-\Delta, \Delta)]} \left\{ \frac{(c_A^2 - 1)}{2} \frac{E[D]}{\Delta + E[U]} \left[\int_0^\infty \left\{ \int_x^\infty (y-x) dF_{D(0, L_0)}(y) \right. \right. \right. \\
 & \left. \left. \left. - \int_{x-\Delta}^\infty (y-x+\Delta) dF_{U+D(0, L_0)}(y) \right\} dx \right] \right. \\
 & + \frac{E[D]}{\Delta + E[U]} I_1 \\
 & \left. + \frac{1}{2(\Delta + E[U])} \int_0^\infty \int_{x-\Delta}^\infty (y-x+\Delta)^2 dF_{D(0, L_0)}(y) \right\}
 \end{aligned}$$

$$\begin{aligned}
 E[X_\gamma^2] = & \frac{1}{E[B(0, L_0)]} \left\{ \frac{(c_A^2 - 1)}{2} \frac{E[D]}{\Delta + E[U]} \left[\int_0^\infty \left\{ \int_x^\infty (Y - x) dF_{D(0, L_0)}(Y) \right. \right. \right. \\
 & \left. \left. \left. - \int_{x-\Delta}^\infty (Y - x + \Delta) dF_{U+D(0, L_0)}(Y) \right\} \right] \right. \\
 & + \frac{E[D]}{\Delta + E[U]} I_2 \\
 & \left. + \frac{1}{2(\Delta + E[U])} \int_0^\infty x \int_{x-\Delta}^\infty (Y - x + \Delta)^2 dF_{D(0, L_0)}(Y) \right\}
 \end{aligned}$$

Elaborating the integrals we obtain after some algebra

$$\begin{aligned}
 E[X_\gamma] = & \frac{1}{E[B(-\Delta, \Delta)]} \left\{ \frac{(c_A^2 - 1)}{2} \frac{E[D]}{(\Delta + E[U])} \left(\frac{E[D^2(0, L_0)]}{2} \right. \right. \\
 & \left. \left. + \frac{E[U+D(0, L_0)]^2}{2} \right. \right. \\
 & \left. \left. - \Delta E[U+D(0, L_0)] \right) \right. \\
 & + \frac{E[D]}{\Delta + E[U]} I_1 \\
 & \left. + \frac{1}{2(\Delta + E[U])} \left[\frac{E[D^3(0, L_0)]}{3} + \frac{\Delta^3}{3} + \Delta^2 E[D(0, L_0)] + \Delta E[D^2(0, L_0)] \right] \right\} \\
 & (5.42)
 \end{aligned}$$

$$\begin{aligned}
E[X_\gamma^2] = & \frac{2}{E[B(-\Delta, \Delta)]} \left\{ \frac{(c_A^2 - 1)}{2} \frac{E[D]}{\Delta + E[U]} \left(\frac{E[D^3(0, L_0)]}{6} \right. \right. \\
& + \frac{\Delta^3}{6} + \frac{E[U + D(0, L_0)]}{2} \Delta^2 + \frac{E[(U + D(0, L_0))^2]}{2} \Delta \\
& + \frac{E[(U + D(0, L_0))^2]}{6} \\
& + \frac{E[D]}{\Delta + E[U]} I_2 \\
& + \frac{1}{2(\Delta + E[U])} \left\{ \frac{\Delta^4}{12} + \frac{E[D(0, L_0)]\Delta^3}{3} \right. \\
& + \frac{E[D^2(0, L_0)]\Delta^2}{2} + \frac{E[D^3(0, L_0)]\Delta}{3} \\
& + \left. \left. \frac{E[D^4(0, L_0)]}{12} \right\} \right\}
\end{aligned} \tag{5.43}$$

Note that the term $E[D^3(0, L_0)]$ vanishes in $E[X_\gamma]$, since this term occurs in (5.40) and (5.42) with opposite signs. The same holds for the term $E[D^4(0, L_0)]$ in $E[X_\gamma^2]$. We emphasize that we do not use explicit expressions for higher moments of $D(0, L_0)$ than the first two, since we assume that $D(0, L_0)$ is gamma distributed. The same holds for the higher moments of D .

Though the above expressions for $E[X_\gamma]$ and $E[X_\gamma^2]$ are quite complicated, it is a routine matter to apply them in computer software. Note also that $E[X_\gamma]$ and $E[X_\gamma^2]$ only depend on Δ and not on s , which may offer computational advantages, when calculating the physical stock for several values of s with fixed Δ .

The PDF-method now prescribes to fit a gamma-distribution $\hat{\gamma}(\cdot)$ to $E[X_\gamma]$ and $E[X_\gamma^2]$. Then we claim that

$$E[B(x, \Delta)] \approx E[B(-\Delta, \Delta)](1 - \hat{\gamma}(x + \Delta)) \quad x \geq -\Delta \tag{5.44}$$

This is our approximation for $B(x, \Delta)$ for $x \geq -\Delta$. Then we can find an approximation for $E[X^+(s, \Delta)]$ from (5.31).

Thus we suggest the following approximations for $E[X^+(s, \Delta)]$.

$$E[X^+(s, \Delta)] = \begin{cases} s - \frac{E[D]E[L]}{E[A]} + \frac{(c_a^2 - 1)}{2} \left[\int_s^\infty (y-s) dF_{D(0, L_0]}(y) - \int_s^\infty (y-s) dF_{\Delta + D(0, L_0]}(y) \right] \\ + \int_s^\infty (y-s) dF_{D(0, L_0]}(y) & \Delta = 0 \\ s - \frac{E[D]E[L]}{E[A]} + E[U] + \frac{\Delta^2 - E[U^2]}{2(\Delta + E[U])} \\ + E[B(-\Delta, \Delta)](1 - \hat{\gamma}(s + \Delta)) & \Delta > 0 \end{cases} \quad (5.45)$$

To complete the analysis we also give the expressions for $E[B(s, \Delta)]$ for the cases of $\Delta > 0$ and $\Delta = 0$, and $s \geq -\Delta$.

$$E[B(s, \Delta)] \simeq \begin{cases} \frac{E[L]E[D]}{E[A]}(1 - \hat{\gamma}(s)) & \Delta = 0 \\ E[B(-\Delta, \Delta)](1 - \hat{\gamma}(s + \Delta)) & \Delta > 0 \end{cases} \quad (5.46)$$

Here $\hat{\gamma}(\cdot)$ is the gamma distribution with first and second moment $E[X_1]$ and $E[X_1^2]$, respectively, given by

$$E[X_1] = \frac{E[A]}{E[L]E[D]} \left\{ \frac{E[D^2(0, L_0)]}{2} - \frac{(c_a^2 - 1)}{2} \left[\frac{E[D^2]}{2} + E[D]E[D(0, L_0)] \right] \right\} \quad (5.47)$$

$$E[X_1^2] = \frac{E[A]}{E[L]E[D]} \left\{ \frac{E[D^3(0, L_0)]}{3} - \frac{(c_a^2 - 1)}{2} \left[\frac{E[D^3]}{3} + \frac{E[D(0, L_0)]E[D^2]}{2} + \frac{E[D^2(0, L_0)]E[D]}{2} \right] \right\} \quad (5.48)$$

Note that $E[B(0,0)]$ equals $E[0]$, the average amount on order. This is intuitively clear, since for the case of $\Delta=0$ each individual demand is ordered at the supplier and because of $s=0$ each demand is backordered and fulfilled after the order is received. Therefore the backlog and the outstanding orders are equivalent.

In the literature usually the mean physical stock is approximated by an interpolation formula,

$$E[X^*(s,\Delta)] = \frac{1}{2} (E(\text{maximum net stock in a cycle}) \\ + E(\text{minimum net stock in a cycle}))$$

For the (s,S) -model this approach yields

$$E[X^*(s,\Delta)] = \frac{1}{2} (s+\Delta - E[D(0,L_0)] + s - E[U] - E[D(0,L_0)]) \\ = s + \frac{\Delta}{2} - E[D(0,L_0)] - \frac{E[U]}{2} \quad (5.49)$$

Note that we included the undershoot, which is often ignored. If we ignore the backlog we also find such a simple approximation for $E[X^*(s,\Delta)]$ from (5.45).

$$E[X^*(s,\Delta)] \approx s + \frac{\Delta}{2} - E[D(0,L_0)] - \frac{E[U]}{2} \\ + E[U] - \frac{E[U^2] - E^2[U]}{2(\Delta + E[U])} \\ + \frac{(c_A^2 - 1)}{2} E[D] \quad (5.50)$$

Note that (5.49) and (5.50) only coincide when U is negligible and the arrival process is a Poisson process.

Typically for Δ large (5.49) yields an underestimate of the mean physical stock.

5.3. Cost considerations

As for the (b, Q) -model we assume that the holding cost per s.k.u. per time unit equals h and the penalty cost per unit short per time unit equals p . The fixed order cost equals K . We define $g(s, \Delta)$ by

$g(s, \Delta) :=$ the mean total cost per time unit associated with the $(s, s+\Delta)$ -policy

i.e.

$$g(s, \Delta) = hE[X^+(s, \Delta)] + pE[B(s, \Delta)] + K/E[\sigma_1] \quad (5.51)$$

We want to solve for (s^*, Δ^*) satisfying

$$g(s^*, \Delta^*) \leq g(s, \Delta) \quad \forall (s, \Delta) \quad (5.52)$$

We can evaluate the Kuhn-Tucher conditions to find that a necessary condition for (5.51) to hold is that

$$\hat{p}_1(s^*, \Delta^*) = \frac{p}{p+h} \quad (5.53)$$

This is identical to condition (4.54). Apparently this is a structural result for inventory management models with the above cost structure.

Again the optimization procedure consists of a one-dimensional search for Δ^* given $s(\Delta)$ derived from (5.53); i.e.

$$\min_{\Delta} g(s(\Delta), \Delta) \quad (5.54)$$

where $s(\Delta)$ satisfies

$$\hat{P}_1(s(\Delta), \Delta) = \frac{P}{p+h} \quad (5.55)$$

Equation (5.55) is routinely solved using the PDF-method as given by (5.23) and (5.24). The minimization procedure associated with (5.54) is a routine matter due to the convexity of $g(s(.), .)$.

A short-cut method, which applied quite well in practice, is to take Δ^* equal to the Economic Order Quantity.

REFERENCES

1. Abramowitz, M and I.A. Stegun, 1965, Handbook of mathematical functions, Dover, New York.
2. Burgin, T., 1975, The gamma distribution and inventory control, Oper.Res. Quarterly 26, 507-525.
3. Chambers, J.C., Mullick, S.K. and Smith, D.D., 1971, How to choose the right forecasting technique, Harvard Business Review, July-August, 45-74.
4. Cinlar, E.H., 1975, Introduction to stochastic processes, Prentice-Hall, Englewood Cliffs, New Jersey.
5. De Kok, A.G., 1987, Production-inventory control models: Algorithms and approximations, CWI-tract. nr. 30, CWI Amsterdam.
6. De Kok, A.G., 1990, Hierarchical production planning for consumer goods, European Journal of Operational Research 45, 55-69.
7. De Kok, A.G. and Van der Heijden, M.C., 1990, Approximating performance characteristics for the (R,S) inventory system as a part of a logistic network, CQM-note 82, Centre for Quantitative Methods, Philips Electronics, Eindhoven (submitted for publication).
8. De Kok, A.G., 1991, A simple and robust algorithm for computing inventory control policies, CQM-note 83, Centre for Quantitative Methods, Philips Electronics, Eindhoven (submitted for publication).
9. Hadley, G. and Whitin T.M., 1963, Analysis of inventory systems, Prentice-Hall, Englewood Cliffs, New Jersey.
10. IBM Corporation, 1972, Basic principles of wholesale-IMPACT-Inventory Management Program and Control Techniques, Second Edition, GE20-8105-1, White Plains, New York.
11. Press, W.H., Flannery, B.P., Tenkolsky, S.A. and Vetterling, W.I., 1986, Numerical recipes, the art of scientific computing, Cambridge University Press, Cambridge.
12. Ross, S.M., 1970, Applied probability models with optimization applications, Holden-Day, San Francisco.
13. Silver, E.A. and Peterson, R. 1985, Decision systems for inventory management and production planning, Wiley, New York.

14. Tijms, H.C., 1986, Stochastic modelling and analysis - a computational approach, Wiley, Chichester.
15. Van der Veen, B., 1981, Safety stocks - an example of theory and practice in O.R., European Journal of Operational Research 6, 367-371.

IN 1990 REEDS VERSCHENEN

- 419 Bertrand Melenberg, Rob Alessie
A method to construct moments in the multi-good life cycle consumption model
- 420 J. Kriens
On the differentiability of the set of efficient (μ, σ^2) combinations in the Markowitz portfolio selection method
- 421 Steffen Jørgensen, Peter M. Kort
Optimal dynamic investment policies under concave-convex adjustment costs
- 422 J.P.C. Blanc
Cyclic polling systems: limited service versus Bernoulli schedules
- 423 M.H.C. Paardekooper
Parallel normreducing transformations for the algebraic eigenvalue problem
- 424 Hans Gremmen
On the political (ir)relevance of classical customs union theory
- 425 Ed Nijssen
Marketingstrategie in Machtsperspectief
- 426 Jack P.C. Kleijnen
Regression Metamodels for Simulation with Common Random Numbers: Comparison of Techniques
- 427 Harry H. Tigelaar
The correlation structure of stationary bilinear processes
- 428 Drs. C.H. Veld en Drs. A.H.F. Verboven
De waardering van aandelenwarrants en langlopende call-opties
- 429 Theo van de Klundert en Anton B. van Schaik
Liquidity Constraints and the Keynesian Corridor
- 430 Gert Nieuwenhuis
Central limit theorems for sequences with $m(n)$ -dependent main part
- 431 Hans J. Gremmen
Macro-Economic Implications of Profit Optimizing Investment Behaviour
- 432 J.M. Schumacher
System-Theoretic Trends in Econometrics
- 433 Peter M. Kort, Paul M.J.J. van Loon, Mikuláš Luptacik
Optimal Dynamic Environmental Policies of a Profit Maximizing Firm
- 434 Raymond Gradus
Optimal Dynamic Profit Taxation: The Derivation of Feedback Stackelberg Equilibria

- 435 Jack P.C. Kleijnen
Statistics and Deterministic Simulation Models: Why Not?
- 436 M.J.G. van Eijs, R.J.M. Heuts, J.P.C. Kleijnen
Analysis and comparison of two strategies for multi-item inventory systems with joint replenishment costs
- 437 Jan A. Weststrate
Waiting times in a two-queue model with exhaustive and Bernoulli service
- 438 Alfons Daems
Typologie van non-profit organisaties
- 439 Drs. C.H. Veld en Drs. J. Grazell
Motieven voor de uitgifte van converteerbare obligatieleningen en warrantobligatieleningen
- 440 Jack P.C. Kleijnen
Sensitivity analysis of simulation experiments: regression analysis and statistical design
- 441 C.H. Veld en A.H.F. Verboven
De waardering van conversierechten van Nederlandse converteerbare obligaties
- 442 Drs. C.H. Veld en Drs. P.J.W. Duffhues
Verslaggevingsaspecten van aandelenwarrants
- 443 Jack P.C. Kleijnen and Ben Annink
Vector computers, Monte Carlo simulation, and regression analysis: an introduction
- 444 Alfons Daems
"Non-market failures": Imperfecties in de budgetsector
- 445 J.P.C. Blanc
The power-series algorithm applied to cyclic polling systems
- 446 L.W.G. Strijbosch and R.M.J. Heuts
Modelling (s,Q) inventory systems: parametric versus non-parametric approximations for the lead time demand distribution
- 447 Jack P.C. Kleijnen
Supercomputers for Monte Carlo simulation: cross-validation versus Rao's test in multivariate regression
- 448 Jack P.C. Kleijnen, Greet van Ham and Jan Rotmans
Techniques for sensitivity analysis of simulation models: a case study of the CO₂ greenhouse effect
- 449 Harrie A.A. Verbon and Marijn J.M. Verhoeven
Decision-making on pension schemes: expectation-formation under demographic change

- 450 Drs. W. Reijnders en Drs. P. Verstappen
Logistiek management marketinginstrument van de jaren negentig
- 451 Alfons J. Daems
Budgeting the non-profit organization
An agency theoretic approach
- 452 W.H. Haemers, D.G. Higman, S.A. Hobart
Strongly regular graphs induced by polarities of symmetric designs
- 453 M.J.G. van Eijs
Two notes on the joint replenishment problem under constant demand
- 454 B.B. van der Genugten
Iterated WLS using residuals for improved efficiency in the linear model with completely unknown heteroskedasticity
- 455 F.A. van der Duyn Schouten and S.G. Vanneste
Two Simple Control Policies for a Multicomponent Maintenance System
- 456 Geert J. Almekinders and Sylvester C.W. Eijffinger
Objectives and effectiveness of foreign exchange market intervention
A survey of the empirical literature
- 457 Saskia Oortwijn, Peter Borm, Hans Keiding and Stef Tijs
Extensions of the τ -value to NTU-games
- 458 Willem H. Haemers, Christopher Parker, Vera Pless and Vladimir D. Tonchev
A design and a code invariant under the simple group Co_3
- 459 J.P.C. Blanc
Performance evaluation of polling systems by means of the power-series algorithm
- 460 Leo W.G. Strijbosch, Arno G.M. van Doorne, Willem J. Selen
A simplified MOLP algorithm: The MOLP-S procedure
- 461 Arie Kapteyn and Aart de Zeeuw
Changing incentives for economic research in The Netherlands
- 462 W. Spanjers
Equilibrium with co-ordination and exchange institutions: A comment
- 463 Sylvester Eijffinger and Adrian van Rixtel
The Japanese financial system and monetary policy: A descriptive review
- 464 Hans Kremers and Dolf Talman
A new algorithm for the linear complementarity problem allowing for an arbitrary starting point
- 465 René van den Brink, Robert P. Gilles
A social power index for hierarchically structured populations of economic agents

IN 1991 REEDS VERSCHENEN

- 466 Prof.Dr. Th.C.M.J. van de Klundert - Prof.Dr. A.B.T.M. van Schaik
Economische groei in Nederland in een internationaal perspectief
- 467 Dr. Sylvester C.W. Eijffinger
The convergence of monetary policy - Germany and France as an example
- 468 E. Nijssen
Strategisch gedrag, planning en prestatie. Een inductieve studie binnen de computerbranche
- 469 Anne van den Nouweland, Peter Borm, Guillermo Owen and Stef Tijs
Cost allocation and communication
- 470 Drs. J. Grazell en Drs. C.H. Veld
Motieven voor de uitgifte van converteerbare obligatieleningen en warrant-obligatieleningen: een agency-theoretische benadering
- 471 P.C. van Batenburg, J. Kriens, W.M. Lammerts van Bueren and R.H. Veenstra
Audit Assurance Model and Bayesian Discovery Sampling
- 472 Marcel Kerkhofs
Identification and Estimation of Household Production Models
- 473 Robert P. Gilles, Guillermo Owen, René van den Brink
Games with Permission Structures: The Conjunctive Approach
- 474 Jack P.C. Kleijnen
Sensitivity Analysis of Simulation Experiments: Tutorial on Regression Analysis and Statistical Design
- 475 C.P.M. van Hoesel
An $O(n \log n)$ algorithm for the two-machine flow shop problem with controllable machine speeds
- 476 Stephan G. Vanneste
A Markov Model for Opportunity Maintenance
- 477 F.A. van der Duyn Schouten, M.J.G. van Eijs, R.M.J. Heuts
Coordinated replenishment systems with discount opportunities
- 478 A. van den Nouweland, J. Potters, S. Tijs and J. Zarzuelo
Cores and related solution concepts for multi-choice games
- 479 Drs. C.H. Veld
Warrant pricing: a review of theoretical and empirical research
- 480 E. Nijssen
De Miles and Snow-typologie: Een exploratieve studie in de meubel-
branche
- 481 Harry G. Barkema
Are managers indeed motivated by their bonuses?

- 482 Jacob C. Engwerda, André C.M. Ran, Arie L. Rijkeboer
Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation $X + A^T X^{-1} A = I$
- 483 Peter M. Kort
A dynamic model of the firm with uncertain earnings and adjustment costs
- 484 Raymond H.J.M. Gradus, Peter M. Kort
Optimal taxation on profit and pollution within a macroeconomic framework
- 485 René van den Brink, Robert P. Gilles
Axiomatizations of the Conjunctive Permission Value for Games with Permission Structures
- 486 A.E. Brouwer & W.H. Haemers
The Gewirtz graph - an exercise in the theory of graph spectra
- 487 Pim Adang, Bertrand Melenberg
Intratemporal uncertainty in the multi-good life cycle consumption model: motivation and application
- 488 J.H.J. Roemen
The long term elasticity of the milk supply with respect to the milk price in the Netherlands in the period 1969-1984
- 489 Herbert Hamers
The Shapley-Entrance Game
- 490 Rezaul Kabir and Theo Vermaelen
Insider trading restrictions and the stock market
- 491 Piet A. Verheyen
The economic explanation of the jump of the co-state variable
- 492 Drs. F.L.J.W. Manders en Dr. J.A.C. de Haan
De organisatorische aspecten bij systeemontwikkeling een beschouwing op besturing en verandering
- 493 Paul C. van Batenburg and J. Kriens
Applications of statistical methods and techniques to auditing and accounting
- 494 Ruud T. Frambach
The diffusion of innovations: the influence of supply-side factors
- 495 J.H.J. Roemen
A decision rule for the (des)investments in the dairy cow stock
- 496 Hans Kremers and Dolf Talman
An SLSP algorithm to compute an equilibrium in an economy with linear production technologies

- 497 L.W.G. Strijbosch and R.M.J. Heuts
Investigating several alternatives for estimating the compound lead
time demand in an (s,Q) inventory model
- 498 Bert Bettonvil and Jack P.C. Kleijnen
Identifying the important factors in simulation models with many
factors
- 499 Drs. H.C.A. Roest, Drs. F.L. Tijssen
Beheersing van het kwaliteitsperceptieproces bij diensten door middel
van keurmerken
- 500 B.B. van der Genugten
Density of the F-statistic in the linear model with arbitrarily
normal distributed errors
- 501 Harry Barkema and Sytse Douma
The direction, mode and location of corporate expansions
- 502 Gert Nieuwenhuis
Bridging the gap between a stationary point process and its Palm
distribution
- 503 Chris Veld
Motives for the use of equity-warrants by Dutch companies
- 504 Pieter K. Jagersma
Een etiologie van horizontale internationale ondernemingsexpansie
- 505 B. Kaper
On M-functions and their application to input-output models
- 506 A.B.T.M. van Schaik
Productiviteit en Arbeidsparticipatie
- 507 Peter Borm, Anne van den Nouweland and Stef Tijs
Cooperation and communication restrictions: a survey
- 508 Willy Spanjers, Robert P. Gilles, Pieter H.M. Ruys
Hierarchical trade and downstream information
- 509 Martijn P. Tummers
The Effect of Systematic Misperception of Income on the Subjective
Poverty Line
- 510 A.G. de Kok
Basics of Inventory Management: Part 1
Renewal theoretic background
- 511 J.P.C. Blanc, F.A. van der Duyn Schouten, B. Pourbabai
Optimizing flow rates in a queueing network with side constraints
- 512 R. Peeters
On Coloring j-Unit Sphere Graphs

- 513 Drs. J. Dagevos, Drs. L. Oerlemans, Dr. F. Boekema
Regional economic policy, economic technological innovation and networks
- 514 Erwin van der Krabben
Het functioneren van stedelijke onroerend-goed-markten in Nederland - een theoretisch kader
- 515 Drs. E. Schaling
European central bank independence and inflation persistence
- 516 Peter M. Kort
Optimal abatement policies within a stochastic dynamic model of the firm
- 517 Pim Adang
Expenditure versus consumption in the multi-good life cycle consumption model
- 518 Pim Adang
Large, infrequent consumption in the multi-good life cycle consumption model
- 519 Raymond Gradus, Sjak Smulders
Pollution and Endogenous Growth
- 520 Raymond Gradus en Hugo Keuzenkamp
Arbeidsongeschiktheid, subjectief ziektegevoel en collectief belang
- 521 A.G. de Kok
Basics of inventory management: Part 2
The (R,S)-model
- 522 A.G. de Kok
Basics of inventory management: Part 3
The (b,Q)-model

Bibliotheek K. U. Brabant



17 000 01066348 3